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Hedonic model with discrete consumer heterogeneity and horizontal differentiated housing*

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Abstract

This paper investigates how the hedonic equilibrium is modified when discrete consumer heterogeneity with horizontal differentiated housing supply is assumed. Our results are threefold. First, discrete consumer heterogeneity leads to a segmentation of the hedonic price function at equilibrium and the discontinuity of the implicit price of environmental quality on the borders of the segments. Second, we demonstrate that horizontal differentiation may lead to a partial sorting of consumer demand for housing attributes at hedonic equilibrium. Finally, we show that the discrete consumer heterogeneity with horizontal differentiation can lead to modification of welfare assessment related to changes in environmental quality.

JEL classification : R21, R31, Q51
Keywords : Hedonic model, Discrete consumer heterogeneity, Horizontal differentiation, Locational choice

Introduction

The hedonic model analyzes consumer choice of differentiated goods. It permits the estimation of the value of the non-market attributes of the differentiated good. The most common empirical application of the hedonic model is in environmental economics when evaluating local amenities and/ nuisances (see for example, Boyle and Kiel, 2001; Simons and Saginor, 2006). The goal of environmental hedonic valuation is to use the price function of a differentiated good (usually housing) to identify individual demand for amenities or, at least, marginal willingness to pay (MWTP) for marginal improvement in environmental quality. The method makes two main assumptions: that consumers assign a value to the environmental quality and that this value is

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reflected in housing price. Achieving this goal depends on assumptions about consumers' and housing heterogeneity.

In the present paper we build a theoretical hedonic model to investigate the consequences of exogenous segmentation of consumers in the presence of horizontal differentiated housing.[1] We adopt the same structure as the main theoretical approaches of hedonic modeling, i.e. a continuum of heterogeneous consumers, each buying a differentiated good (housing) represented by heterogeneous attributes (including environmental quality). However, our model assumes a discrete heterogeneity on the demand side and continuous heterogeneity on the supply side.[2] Horizontal differentiation generally means that different consumers, or in the present study groups of consumers, have different preferences for the same house (i.e. with the same attributes). We make a distinction between two cases of the horizontal differentiation. In the first case, the horizontal differentiation only concerns the environmental quality of a dwelling: every group has differing preferences for environmental quality of housing but share the same preferences for other housing attributes. In the second case, the groups have differing preferences for all housing attributes. The two cases lead to different consequences for the hedonic equilibrium, the hedonic price function, the implicit prices of housing attributes and thus for the environmental hedonic valuation.

The existing literature exhibits two main theoretical approaches to hedonic modeling which differ in their assumptions regarding the nature of heterogeneity. The "Traditional" Hedonic Model, developed by Rosen (1974), analyzes the properties of the hedonic price function which arises at equilibrium when a continuum of heterogeneous consumers choose among a continuum of differentiated goods. Consumer heterogeneity parameters and the attributes of goods are

[1]We consider here a partial equilibrium setting. To complete the model in a general equilibrium setting would require the existence of a labor market and a land market. A joint housing and labor markets setting in a sorting model is investigated by Kuminoff (2011). Hidano (2010) analyses different types of capitalization hypothesis and their consideration in hedonic modeling. The general equilibrium analysis is outside the scope of this paper but could constitute a path for further investigations.

[2]In contrast with Ekeland (2010), the question here is to study the implications of the groupwise consumer heterogeneity and horizontal differentiation on supply side, to the environmental hedonic valuation. It involves studying a general case of the utility function (non-separable), and imposing additional restrictions about the structure of supply and demand to be consistent with the empirical environmental hedonic literature.

supposed to be continuously distributed. Rosen's model leads to the famous two-stage estimation procedure: in the first stage the hedonic price function is estimated and the marginal willingness to pay for the environmental attribute is calculated, in the second stage the environmental demand is identified. The second approach, called "New Hedonic" or "Sorting", which developed following a seminal article by Tiebout (1956), concerns the provision of local public goods.[3] Sorting models assume a continuous income and/or taste heterogeneity of consumers who choose a location from a finite number of possible communities. These models therefore assume a discrete heterogeneity of environmental quality on the supply side.[4] In sorting models, the price function does not identify the environmental demand or the MWTP. Given a utility function, one can characterize the market sorting of the consumers between communities and analyze how it affects decision making. Neither of the two approaches address the case of discrete consumer heterogeneity.

Discrete heterogeneity means that there is a finite number of different consumer groups (even if there is always a continuum of consumers). Such heterogeneity may have various origins: borrowing constraints (for example, depending the temporary or permanent nature of employment the ability of an agent to borrow can differ), family structure (single, family without children, family with young children, etc), job market position (active or retired, working locally or commuting, etc). The different sources of this discrete consumer heterogeneity may lead to different types of housing differentiation. When consumers agree about the ranking of housing, this is vertical differentiation. Income heterogeneity is a "classical" source of vertical differentiation (see, for example, Shaked and Sutton, 1981). When different consumers have different preferences about the same housing, there is horizontal differentiation. In all cases, the consequences of discrete consumer heterogeneity are important for hedonic equilibrium. *Ex ante* all consumers have the ability to distinguish between all housing in the relevant market. But, *ex post* con-

[3]See a recent review of the sorting literature by Kuminoff et al. (2010) for a comprehensive analysis of the use of sorting models in environmental policy evaluation.

[4]Environmental quality, such as air quality, can be thought of as homogeneous within one community and heterogenous between different communities if the size of each community is relatively small (see, for example, Kuminoff, 2009).

sumers reveal their type through their consumption choices, and consumer types sort themselves into different segments of the market. Thus, the hedonic price function which characterizes the equilibrium of the whole market, is dependent on individual characteristics and the definition of the hedonic equilibrium is therefore modified. Furthermore, discrete consumer heterogeneity may have important implications when public policies modifying the attributes of all or a subset of housing (e.g. projects to improve environmental quality (noise or air pollution diminution), urban renewal projects, or transport policies to improve accessibility) are evaluated. These policies may have a sizable impact on housing supply, which may shift the whole hedonic price function. When the hedonic price function is segmented and each segment corresponds to one group, moving the hedonic price function may lead to further implications for the overall welfare impact of the policy.

Baudry and Maslianskaïa-Pautrel (2012) develop a hedonic model where a discrete heterogeneity of consumers is assumed with vertically differentiated housing. The authors demonstrate that such a discrete heterogeneity with vertical differentiation can lead to the segmentation of the hedonic price function. This framework goes beyond Rosen's model and thus it questions the use of the two-stage estimation procedure. The authors also show that the impact of segmentation is important when the number of consumer groups is small.

However, the Baudry and Maslianskaïa-Pautrel (2012) model does not take into account the cases where a number of population groups coexist on the same territory and different groups have different preferences about the same housing attributes, namely environmental quality. Generally such cases can arise when the territory is highly heterogeneous, for example when it contains rich natural resources on the one hand and significant industrial development or thriving touristic and recreational areas on the other hand (e.g. coastal or mountainous regions). The French Loire estuary area is a good example with three well-defined population groups: residents working in the area (around 45-60%); residents working outside the area and choosing the area for its environmental quality and its proximity to employment, such as in a regional capital, Nantes in this case (around 20-25%); and retirees choosing this area because of its environmental quality (around 20-30%).[5] With such distinct groups in the same area, we would expect housing attribute preferences, especially for the environmental quality, to differ between groups. For example, proximity to a major road network would mean only noise pollution for one group, but for another group this proximity could also mean an amenity, a reduction in time spent commuting to work. Thus the housing supply is horizontally differentiated, in the spirit of Lancaster (1966), i.e. the aggregation of attributes of the same house generates a different index of housing services for at least two consumers. Further, in these areas, a small number of consumer groups is a reasonable assumption. We can observe such a heterogeneity with a small number of consumer groups in other areas. Lipscomb and Farmer (2005) find three types of households within a single neighborhood: "low income students" who rent their accomodation, "young adults" who rent a larger house or purchase less expensive homes and "more established homeowners".

Similarly to Baudry and Maslianskaïa-Pautrel (2012), we assume a discrete heterogeneity of consumers. But our current model differs significantly because horizontal differentiation leads to different consumer behavior, modifying the existence of equilibrium and its implications. Our main results are threefold. First, the groupwise consumers heterogeneity leads to a segmentation of the hedonic price function in the housing market equilibrium. Hedonic price function is continuous, but built of different segments which correspond to different consumer groups. Therefore, the segmentation of the hedonic price function implies a discontinuity of the implicit price of the environmental quality of housing on the borders of the segments. This means that the second stage of the "usual" hedonic estimation procedure (where the implicit price function is assumed to be continuous) cannot be used.[6] Thus, our result extends the Baudry and Maslianskaïa-Pautrel (2012) finding with vertical differentiated housing, and highlights the role

[5]Data sources: census track data, *INSEE, RP2007, Principal and complementary operations.*

[6]This result calls for further investigations into empirical implications which are outside the scope of this paper. In this field some connections can be made with the nonparametric approach used by Ekeland et al. (2004) and Heckman et al. (2010). They stress the importance of nonparametric estimation of the hedonic price function. Such nonparametric models would be able to handle the discontinuities in the implicit prices.

played by discrete heterogeneity on the demand side of hedonic equilibrium.

The second result is the sorting of consumer demand. We demonstrate that the segmentation of the hedonic price function may lead to partial sorting of the demand for housing attributes by consumer groups with a horizontal differentiation of housing supply (namely, when the horizontal differentiation concerns all housing attributes). This result differs from the case of vertical differentiated housing supply described in Baudry and Maslianskaïa-Pautrel (2012). It can also be viewed as dual to the partial two-dimensional stratification by consumer income and preferences in sorting models (see Epple and Platt, 1998).

Finally, the third result concerns consumer surplus in equilibrium. With fixed supply and short-run assumptions, a certain number of the sellers extract the overall consumer surplus of one or more groups, but not necessarily from the group with the lowest preferences for housing attributes. Thus the segmentation affects the welfare impact of changes in environmental quality.

Section 1 presents the general framework of the model. Section 2 derives equilibrium conditions. And section 3 discusses the principal results of the model and their implications.

1 General framework

In the economy, there are two consumption goods: housing and a numeraire. Housing is characterized by two attributes, S, the intrinsic attribute, and Q, the environmental attribute, aggregated into a housing service index H: $H = h(Q, S)$.[7] As in the Rosen (1974) model, the housing price, P, depends on these attributes: $P = P(Q, S)$.[8]

[7] In the real market, houses have many different structural characteristics (e.g. bedrooms, bathrooms, square footage, lot size). The scalar S, representing the intrinsic attribute of the dwelling, can be viewed as a theoretically consistent index of "physical housing services". This idea dates back to the work on price indexes by Samuelson and Swamy (1974) and is used in the sorting model by Sieg et al. (2002). In sorting models, the function of housing service is supposed to be separable between a "quantitative" index of intrinsic characteristics of dwelling S and a "qualitative" index of its environmental and neighborhood characteristics Q (Kuminoff, 2009).

[8] In this respect, our model differs from sorting models, where only the qualitative environmental attribute Q determines the price of housing (cf. Kuminoff, 2009). In our case, the price depends on both the intrinsic and environmental characteristics of housing.

1.1 Consumer behavior

We assume a discrete (group-wise) heterogeneity of households: a finite number N of consumer types who choose continuous combinations of housing characteristics. The population in each group is equal to η_i (i group index) and the entire population is normalized to 1: $\sum_{i=1}^{N} \eta_i = 1$.

Household preferences are homogenous within each group. Households in different groups are heterogeneous in their taste, β, for housing attributes. This means that the housing service index depend on the β_i of each group, and can be given as: $H = h(Q, S|\beta_i)$.

We assume the function $h(\cdot)$ belongs to the class $\mathcal{C}^1$, is increasing, concave, and satisfies the Spence-Mirrlees single crossing condition in β:[9]

$$\frac{\partial}{\partial \beta} \left[\frac{\partial H / \partial Q}{\partial H / \partial S} \right] > 0, \ \forall \beta. \tag{$H0$}$$

We assume that groups are positioned in increasing order for the parameter β: $\beta_1 < ... < \beta_N$.

Following the Rosen's model, we consider an individual from group i using her income R to purchase the numeraire good, denoted by X, and a dwelling, characterized by its level of service $H = h(Q, S|\beta_i)$. A household's utility (corresponding to the group's utility function) depends on the housing service level and on its consumption of the numeraire. We assume that households are homogeneous in their income as well as their preferences between housing or other consumption.[10] The utility function of a household in a group i is given by $U(h(Q, S|\beta_i), X)$, where β_i is the group's parameter of heterogeneous preferences for housing attributes.[11] We assume that the utility function belongs to the class $\mathcal{C}^2$, is increasing, and is concave in its arguments.

The budget constraint of a household in a group i is given by $X + P(Q, S) = R$. When replacing the consumption of X by the maximum quantity that the household can spend given its

[9] Without loss of generality we assume that MRS_{QS} is increasing in β.

[10] Since the income heterogeneity is rather continuous, if we allow the income heterogeneity we should to deal with conditional distribution of income in each preference group. We prefer to concentrate on one source of heterogeneity to highlight the role of discrete heterogeneity of consumers.

[11] The individual utility depends also on other parameters, common to all consumers in the group. To simplify the notation, this set of parameters is omitted.

income and the price of the housing, we obtain a constrained utility function $V(h(Q,S|\beta_i),P) \equiv U(h(Q,S|\beta_i), R - P(Q,S))$.

We assume that the constrained utility function (and thus the group bid function) satisfies the following Spence-Mirrlees conditions:

$$\forall S, \quad \frac{\partial}{\partial \beta}\left[\frac{\partial V/\partial h \cdot \partial h/\partial Q}{\partial V/\partial P}\right] < 0, \ \forall \beta, \tag{H1}$$

$$\forall Q, \quad \frac{\partial}{\partial \beta}\left[\frac{\partial V/\partial h \cdot \partial h/\partial S}{\partial V/\partial P}\right] > 0, \ \forall \beta \tag{H2}$$

The constrained utility function is used to introduce the fundamental concept of Rosen's bid function.

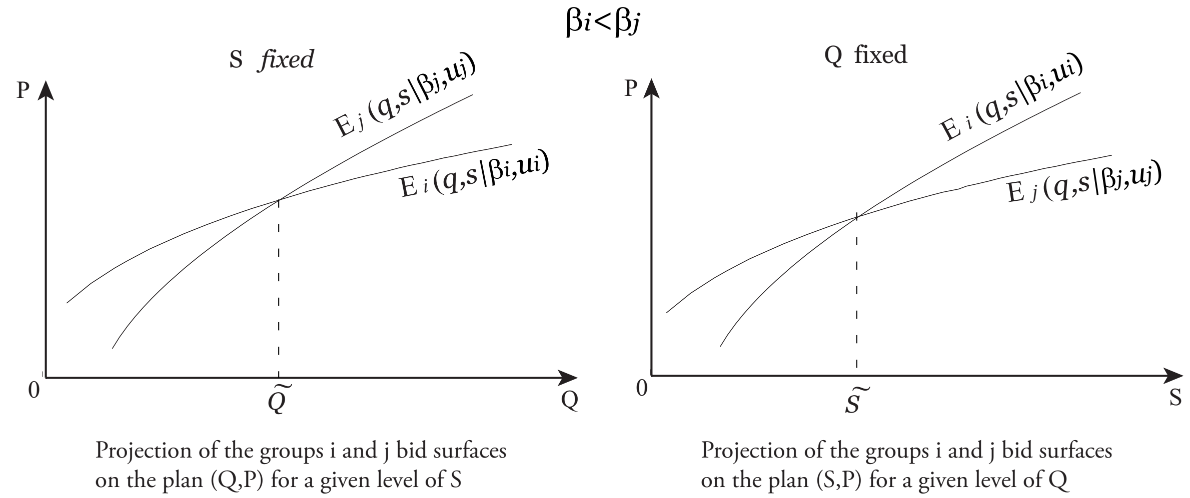
Definition 1. Rosen's bid function *for an individual of group i is defined implicitly by:*

$$V(h(Q,S|\beta_i), E_i^{(k)}(Q,S|\beta_i,u_i^{(k)})) = u_i^{(k)}, \tag{1}$$

where $u_i^{(k)}$, reference utility level, represents the level of utility attained by the individual at her current location k with the index of housing services $H = h(Q^{(k)}, S^{(k)}|\beta_i)$ and the price $P^{(k)}$.

In the space (Q,S,P) the bid surface coincides with the surface of iso constrained utility, therefore the groups' bid surfaces also satisfy the single-crossing properties $(H1)$ and $(H2)$.

Figure 1: Single-crossing property of groups' bid surfaces under $H''1$ and $H''2$



The condition $(H1)$ means that in the space (Q,P) for a given value of S, the slopes of the bid curves increases with β. A larger β corresponds to stronger individual preferences for the environmental quality.

Figure 1 illustrates an economic interpretation of the Spence-Mirrlees conditions $(H1)$ and $(H2)$. Condition $(H2)$ means that in the space (S, P) for a given level of Q, the slopes of the bid curves decreases with β. On the right of the intersection of two bid curves, $\tilde{S}$, consumers in the group with a higher β are willing to pay a lower price for housing with the same level of intrinsic quality $S > \tilde{S}$, and on the left of $\tilde{S}$ those consumers (with higher β) are willing to pay a higher price for housing with the same level of intrinsic quality $S < \tilde{S}$.

Conditions $(H1)$ and $(H2)$ show that housing attributes could be viewed as substitutes for each other and the degree of the substitutability depends on the value of β. In this case, each group has its specific valuation of housing with $Q = 0$ ("*outside the environmental quality housing*") and housing with $S = 0$ ("*outside the intrinsic quality housing*"):

$$E(0, S|\beta_i, u_i^{(k_i)}) \neq E(0, S|\beta_j, u_j^{(k_j)}), \text{ if } \beta_i \neq \beta_j$$

$$E(Q, 0|\beta_i, u_i^{(k_i)}) \neq E(Q, 0|\beta_j, u_j^{(k_j)}), \text{ if } \beta_i \neq \beta_j$$

The Spence-Mirrlees conditions imply that two arbitrary bid surfaces of two different groups have only one intersection line. We define the *intersection line* of two groups i and j as a locus of the points (q, s) such that: $E_i(q, s|\beta_i, u_i^{(k_i)}) = E_j(q, s|\beta_j, u_j^{(k_j)})$. This equation implicitly defines *the equation of the intersection line*, noted $s = g_{ij}(q)$.

Proposition 1. *The intersection line of two bid functions of two groups is a strictly monotonic function on the plane* (Q, S).

Proof. Immediately from single-crossing conditions $H1$ and $H2$. □

Equation (1) defines a family of bid functions for households belonging to the group i: $E_i^{(k)} = E_i^{(k)}(Q, S|\beta_i, u_i^{(k)})$, parameterized by reference utility level $u_i^{(k)}$.

Households in the same group have the same utility function and the same budget constraints. Thus, individual optimizing behavior corresponds to the optimizing behavior of the group to which the household belongs.

The optimization problem of a household's utility maximization, subject to its budget constraint, is equivalent to the problem of its minimization of housing expenditure, i.e.: $\forall i, \quad \min_{u_i^{(k)} \in DU_i} E_i^{(k)}(Q, S|\beta_i, u_i^{(k)}), \quad (Q, S) \in \mathrm{Dh}_i$, where Dh_i is an set (or union of sets) of values (Q, S), characterizing dwellings purchased by consumers in group i. Since the definition of bid function includes the consumer budget constraints (as defined from the constrained utility function), the optimization program of consumers does not have constraints.

1.2 Seller behavior

Following Baudry and Maslianskaïa-Pautrel (2012) we assume the short term case from Rosen's hedonic model, where supply is assumed to be fixed. This assumption is consistent with existent empirical hedonic studies which consider housing stock as constant and, thus, implicitly assume a short term case (see, for example, Palmquist, 2005). Therefore we assume that housing attributes are distributed in $[0; Q_{\max}] \times [0; S_{\max}]$ with a joint density function $\varphi(Q, S)$.[12] This assumption means that sellers do not have control over the Q and S levels of their housing. Thus the seller's "offer" function is reduced to a point.

Given these assumptions regarding the housing supply, the utility maximization problem for each seller s is (Pope, 2006): $\max_{P(Q_s,S_s)} U(h(Q_s, S_s|\beta_s), R + P(Q_s, S_s))$, which is equivalent to the problem of maximizing the selling price of her housing: $\forall s, \forall H_s \quad \max P_s(Q_s, S_s)$

Finally we assume that supply is completed by an *"outside the market" alternative*: $P(0, 0) = 0, \forall \beta$. It is always possible to obtain zero housing characteristics for the price $\mathrm{P} = 0$.[13]

[12] To simplify the notation and calculations, we assume without loss of generality that $Q_{\min} = 0$ and that $S_{\min} = 0$.

[13] The definition assumes that the "outside the market" alternative does not depend on the type of housing differentiation, or the nature of consumers heterogeneity.

1.3 Definition of equilibrium

In the traditional hedonic model, the equilibrium of the housing market is defined when there exists a function $p(Q, S)$ such that the distribution of housing demand and the distribution of housing supply are equal, $H^d(Q, S) = H^s(Q, S), \forall (Q, S)$, and buyers and sellers have an optimal behavior (Rosen, 1974).

In the model with segmentation on the demand side, buyers are in homogeneous groups of non-zero weight, and sellers have to take into consideration buyers' behavior when they define their selling prices. Specifically, two types of constraints must be considered for each buyer: participation and incentive. The first type takes into account the effective participation of the buyer in the market, that is her arbitrage condition between buying a home and choosing the alternative "outside the market". The second constraint represents different alternatives for the consumer in the market and is her arbitrage condition between the purchase of one particular dwelling and the purchase of another dwelling on the market. The hedonic equilibrium of the model with segmentation is therefore defined as:

Definition 2. Hedonic equilibrium of the segmentation model. *With the assumptions of behavior optimization by buyers and sellers, the market equilibrium is reached when market is clearing and the following three conditions are satisfied:*

Condition 1 (Participation constraint): Each buyer b prefers to purchase housing on the market rather than the "outside the market" alternative:

$$\forall b, \quad V(h(Q_b, S_b|\beta_b), P^*(Q_b, S_b)) \geq V(h(0, 0|\beta_b), 0), \quad (Q_b, S_b) \in Dh.$$

Condition 2 (Incentive constraints): For unchangeable prices, each buyer is better off with the housing she buys rather than that bought by other buyers:

$$\forall b, \quad V(h(Q_b, S_b|\beta_b), P^*(Q_b, S_b)) \geq V(h(Q', S'|\beta_b), P^*(Q', S')), \quad \forall (Q', S') \in Dh.$$

Condition 3 (Maximal surplus extraction): None of the sellers, s, are able to find a buyer at a higher price:

$\forall s, \quad P_s(Q_s, S_s) \ = \max_{b,(k):u_b^{(k)} \in DU_b} E_b^k(Q_s, S_s|\beta_b, u_b^k).$

The set of prices P^* resulting from this optimization program characterizes the hedonic price function:

Definition 3. Hedonic price. *For any level of the index H of housing services, the hedonic price gives the highest of the individual bids where each individual bid function is defined in reference to the house bought by the individual and the associated market equilibrium price:*
$\forall (Q, S), \quad P(Q, S) = \max_{u_b^{(k)} \in DU_b} E_b^k(Q, S|\beta_b, u_b^k).$

In the case of horizontal differentiation, the hedonic equilibrium definition 2 becomes:

$$\forall i \in \{1, ..., N\}$$

$$\max_{u^{(i)}} E_i \left(q, s|\beta_i, u_i^{(k)} \right), \ (q, s) \in [Q_{i-1}, Q_i] \times [S_{i-1}, S_i] \tag{2}$$

s.c.

$$V \left(h(q, s|\beta_i), E_i \left(q, s|\beta_i, u_i^{(k)} \right) \right) \geq V(0, 0) \tag{3}$$

$$V \left(h(q, s|\beta_i), E_i \left(q, s|\beta_i, u_i^{(k)} \right) \right) \geq V \left(h(\tilde{q}, \tilde{s}|\beta_i), E_j \left(\tilde{q}, \tilde{s}|\beta_j, u_j^{(k)} \right) \right), \tag{4}$$

$$\forall j \neq i, \ (\tilde{q}, \tilde{s}) \in [Q_{j-1}, Q_j] \times [S_{j-1}, S_j]$$

$$\int_{S_{i-1}}^{S_i} \int_{Q_{i-1}}^{Q_i} \varphi(q, s) \, \mathrm{d}q \, \mathrm{d}s = \eta_i \tag{5}$$

Equation (5) means that the market share of a group i is equal to its weight, i.e. market clearing condition.

Definition 4. *The* Participation constraint *for group i corresponds to the bid surface for which the participation constraint introduced by the inequality (3) is saturated. This bid surface is denoted CP_i and defined implicitly from the equation:*
$V \left(h(q, s|\beta_i), CP_i \left(q, s|\beta_i, u_i^{(CP)} \right) \right) = V(0, 0).$

Definition 5. *The* Incentive constraint with respect to the group j *for group i corresponds to the bid surface of the group i for which its incentive constraint (4) with respect to the group j is saturated. This bid surface is denoted $CI_{ij}^{k_{ij}}(q, s|\beta_i, u_i^{(k_{ij})})$ and defined implicitly by:*

$$V\left(h(q,s|\beta_i), CI_{ij}^{(k_{ij})}\left(q,s|\beta_i,u_i^{(k_{ij})}\right)\right) = V\left(h(\tilde{q},\tilde{s}|\beta_i), E_j^{k_j}\left(\tilde{q},\tilde{s}|\beta_j,u_j^{(k_j)}\right)\right),$$

$(q,s) \in [Q_{i-1}, Q_i] \times [S_{i-1}, S_i], (\tilde{q},\tilde{s}) \in [Q_{j-1}, Q_j] \times [S_{j-1}, S_j]$.

2 Existence of the hedonic equilibrium

It is easy to show that the following equilibrium property is verified:

Proposition 2. *If at equilibrium the individual A from group i buys a house with housing services level $H_A^* = h(Q_A^*, S_A^*|\beta_i)$ for the price $P_A^* = E_i^*(H_A^*|\beta_i, u_i^*)$, thus:*

$$P_j(H_j^*) > E_i^*(H_j^*|\beta_i, u_i^*), \quad \forall j\ : \beta_j \neq \beta_i; \tag{6}$$

$$P_B(H_B^*) = E_j^*(H_B^*|\beta_i, u_i^*), \quad \forall B\ : \beta_B = \beta_A = \beta_i \tag{7}$$

Proof. The proof is straightforward from the definition of the bid function and the fact that the utility function is increasing with respect to its arguments, which implies an increase in the value of the group's utility following the downward displacement of the group bid surface. □

The proposition means that : *i)* for a group i, the prices of houses purchased by consumers of other groups can not be located below the equilibrium bid surface of the group i, and *ii)* the prices of houses purchased by consumers of the group i belong on the same group' equilibrium bid surface.

To analyze the existence of a hedonic equilibrium and its characteristics, we proceed in two steps. The first step is to examine the set of group incentive constraints, the second step is to examine their participation constraints. The existence of the equilibrium depends on group preferences in other words the different Spence-Mirrlees conditions presented above.

2.1 Incentive constraints

Consider the isocost curves of the groups' bid functions in the space (Q,S). By definition of a bid function, the housing services level along the isocost curve is constant. Therefore, the isocost curves coincide with the iso-index curves of the housing services (henceforth " *iso-H curves*"). Given the assumptions in function $H = h(Q,S|\beta)$, the iso-H curves are convex and decreasing in the plane (Q, S) and a higher level of H moves the iso-H curve upward. Therefore, a higher iso-H curve of a group corresponds to a higher level of utility, *ceteris paribus*.

Proposition 3. *At equilibrium, the iso-H curves are ordered in the plane (Q, S) in increasing order of β from the left to the right along the Q axis.*

Proof. See Appendix A.1. □

Theorem 1. *At equilibrium, only the incentive constraints with respect to adjacent groups are saturated.*[14]

Demonstration. See Appendix A.2. □

Definition 6. *The* boundary *between two groups i and j is the curve of intersection of the equilibrium groups bid surfaces.*

Corollary 1. *At equilibrium the hedonic price function is continuous.*

Proof. The result follows from Proposition 2 and the theorem 1. □

At equilibrium the boundary $s = g_{i,i-1}(q)$ between groups i and $i-1$ has the follow property:

Proposition 4. *The boundary between two groups $s = g_{i,i-1}(q)$ is an increasing function in q.*

Proof. Follows from proposition 1. □

To determine the equilibrium, it is required to determine what groups fulfill participation constraints at equilibrium.

[14]The result of the theorem 1 can be viewed as symmetric to the result of the *Boundary indifference* lemma in the Epple and Platt (1998, p. 28, lemma 1) sorting model, in that Epple and Platt study the case of a continuous heterogeneity on the demand side and discrete heterogeneity on the supply side, while we study the case of a discrete heterogeneity on the demand side and a continuous heterogeneity on a supply side

2.2 *Group's participation constraint and iterative construction of equilibrium*

Let study participation constraints of the groups i and j such that $\beta_i < \beta_j$. The curve of the intersection of these participation constraints does not belong to any coordinate plane, and levels of housing services of houses with $Q = 0$ or $S = 0$ are specific for each group: $h(0, s|\beta_i) \neq h(0, s|\beta_j)$ and $h(q, 0|\beta_i) \neq h(q, 0|\beta_j)$.

Let $F_{ij}(q, f_{ij}(q))$ be the intersection line of surfaces CP_i and CP_j. It is defined implicitly by the following equation: $CP_i\left(q, f_{ij}(q)|\beta_i, u_i^{(CP_i)}\right) = CP_j\left(q, f_{ij}(q)|\beta_j, u_j^{(CP_j)}\right)$.

According to the conditions $(H1)$ and $(H2)$, to the left of the curve $F_{ij}(q, f_{ij}(q))$ the participation constraint of group i is located above the participation constraint of group j, and at the right of the intersection curve the order is reversed:

$$CP_i\left(q, s|\beta_i, u_i^{(CP_i)}\right) > CP_j\left(q, s|\beta_j, u_j^{(CP_j)}\right), \quad \text{if } s > f_{ij}(q) \tag{8}$$

$$CP_i\left(q, s|\beta_i, u_i^{(CP_i)}\right) < CP_j\left(q, s|\beta_j, u_j^{(CP_j)}\right), \quad \text{if } s < f_{ij}(q) \tag{9}$$

Proposition 5. *Under assumption $(H1)$ - $(H2)$ and given the independence of the "outside the market" alternative from the preference parameter β, all participation constraints are crossing in the same line: $F_{ij}(q, f_{ij}(q)) \equiv F(q, f(q)),\ \forall i, j \in \{1, 2, ..., I\}$*

Proof. See Appendix A.3. □

The following proposition defines which group saturates its participation constraint at equilibrium.

Proposition 6. *The group i° which saturates its participation constraint at equilibrium can be found from the following condition:*

$$i^\circ = \min_{i \in \{1,...,I\}} i \ : \ \sum_{k=1}^{i} \eta_k \geq \Pi(f(q)), \tag{10}$$

where

$$\Pi(f(q)) = \begin{bmatrix} \int_0^{S_{\max}} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, & \textit{if } S_{\max} \leq Q_{\max}; \\ \int_0^{f(Q_{\max})} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s & \\ + \int_{f(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, & \textit{if } S_{\max} > Q_{\max}. \end{bmatrix}$$

Proof. See Appendix A.4. □

At equilibrium, each group $i < i^\circ$, saturates its incentive constraint with respect to the succeding group passing through their shared boundary. Each group $i > i^\circ$, saturates its incentive constraint with respect to the preceding group passing through their shared boundary. Figure 4 shows different configurations for boundaries between consumer groups. The boundaries can be obtained from conditions of equivalence between housing services index supply and demand distribution:

Thus, on each interior segment, $\forall\, 1 < i < I$, $q \in \left[Q^i_{\min}, Q^i_{\max}\right]$, $s \in \left[S^i_{\min}, S^i_{\max}\right]$:

$$Q^i_{\min} = \begin{bmatrix} 0, \ \text{if } g_{i-1,i}(0) \geq 0, \\ g^{-1}_{i-1,i}(0), \ \text{if } g^{-1}_{i-1,i}(0) \geq 0; \end{bmatrix} \quad Q^i_{\max} = \begin{bmatrix} g^{-1}_{i,i+1}(S_{\max}), & \text{if } g_{i,i+1}(Q_{\max}) \geq S_{\max}, \\ Q_{\max}, & \text{if } g_{i,i+1}(Q_{\max}) < S_{\max}. \end{bmatrix} \tag{11}$$

$$S^i_{\min} = \begin{bmatrix} g_{i,i+1}(0), \ \text{if } g_{i,i+1}(0) \geq 0, \\ 0, \ \text{si } g^{-1}_{i,i+1}(0) \geq 0; \end{bmatrix} \quad S^i_{\max} = \begin{bmatrix} S_{\max}, & \text{if } g_{i-1,i}(Q_{\max}) \geq S_{\max}, \\ g_{i-1,i}(Q_{\max}), & \text{if } g_{i-1,i}(Q_{\max}) < S_{\max}. \end{bmatrix} \tag{12}$$

For the first segment, $i = 1$, $q \in \left[0, Q^1_{\max}\right]$, $s \in \left[S^1_{\min}, S_{\max}\right]$:

$$Q^1_{\max} = \begin{bmatrix} g^{-1}_{1,2}(S_{\max}), & \text{if } g_{1,2}(Q_{\max}) \geq S_{\max}, \\ Q_{\max}, & \text{if } g_{1,2}(Q_{\max}) < S_{\max}. \end{bmatrix} \quad S^1_{\min} = \begin{bmatrix} g_{1,2}(0), \ \text{if } g_{1,2}(0) \geq 0, \\ 0, \ \text{if } g^{-1}_{1,2}(0) \geq 0; \end{bmatrix} \tag{13}$$

For the last segment, $i = I$, $q \in \left[Q^N_{\min}, Q_{\max}\right]$, $s \in \left[0, S^N_{\max}\right]$:

$$Q^N_{\min} = \begin{bmatrix} g^{-1}_{N-1,N}(0), & \text{if } g^0_{N-1,N}(0) \geq 0, \\ 0, & \text{if } g_{N-1,N}(0) \geq 0. \end{bmatrix} \quad S^N_{\max} = \begin{bmatrix} S_{\max}, \ \text{if } g_{N-1,N}(Q_{\max}) \geq S_{\max}, \\ g_{N-1,N}(Q_{\max}), \ \text{if } g_{N-1,N}(Q_{\max}) < S_{\max}; \end{bmatrix} \tag{14}$$

Theorem 2. *Under assumptions (H1) and (H2), there is an equilibrium on the housing market for which the hedonic price function corresponds to the equilibrium groups bid surfaces, $E^*_i(q, s|\beta_i, u^*_i)$, defined through the following equations:*

$$E^*_{i^\circ} = CP_{i^\circ}(q, s|\beta_{i^\circ}, u^*_{i^\circ}),\ \textit{where} \tag{15}$$

$$CP_{i^\circ}(q, s|\beta_{i^\circ}, u^*_{i^\circ})\ :\ V\left(h(q, s|\beta_{i^\circ}), CP_{i^\circ}\left(q, s|\beta_{i^\circ}, u^*_{i^\circ}\right)\right) = V(0, 0),$$

$$i^\circ = \min_{i\in\{1,\dots,I\}} i\ :\ \sum_{k=1}^{i} \eta_k \geq \Pi(f(q)),$$

$$s = f(q)\ :\ CP_{i^\circ}\left(q, f(q)|\beta_{i^\circ}, u^*_{i^\circ}\right) = CP_i\left(q, f(q)|\beta_i, u_i^{(CP)}\right),\ i \in \{1, \dots, I\}\backslash\{i^\circ\} \tag{16}$$

$$\Pi(f(q)) = \begin{cases} \int_0^{S_{\max}} \int_0^{f^{-1}(s)} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s, & \textit{if } S_{\max} \leq Q_{\max}; \\ \int_0^{f(Q_{\max})} \int_0^{f^{-1}(s)} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s & \\ + \int_{f(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s, & \textit{if } S_{\max} > Q_{\max} \end{cases} \tag{17}$$

$\forall\ 1 \leq i < i^\circ$:

$$V\left(h(q, s|\beta_i), E^*_i\left(q, s|\beta_i, u^*_i\right)\right) = V\left(h(q, g_{i,i+1}(q)|\beta_i), E^*_{i+1}\left(q, g_{i,i+1}(q)|\beta_{i+1}, u^*_{i+1}\right)\right), \tag{18}$$

$\forall\ i^\circ < i \leq I$:

$$V\left(h(q, s|\beta_i), E^*_i\left(q, s|\beta_i, u^*_i\right)\right) = V\left(h(q, g_{i-1,i}(q)|\beta_i), E^*_{i-1}\left(q, g_{i-1,i}(q)|\beta_{i-1}, u^*_{i-1}\right)\right), \tag{19}$$

$\forall i,\ s = g_{i,i+1}(q)$:

$$E^*_i(q, g_{i,i+1}(q)|\beta_i, u^*_i) = E^*_{i+1}(q, g_{i,i+1}(q)|\beta_{i+1}, u^{(i+1)}) \tag{20}$$

$$\sum_{k=1}^{i} \Pi_i(g_{i,i+1}(q)) = \sum_{k=1}^{i} \eta_i,\ i \in \{1, \dots, I\},\ \textit{where} \tag{21}$$

$$\sum_{k=1}^{i} \Pi_i(g_{i,i+1}(q)) = \begin{cases} \int_{S^i_{\min}}^{S_{\max}} \int_{Q^i_{\min}}^{g^{-1}_{i,i+1}(s)} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s, & \textit{if } S_{\max} \leq Q_{\max}; \\ \int_{S^i_{\min}}^{g_{i,i+1}(Q_{\max})} \int_{Q^i_{\min}}^{g^{-1}_{i,i+1}(s)} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s & \\ + \int_{g_{i,i+1}(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s, & \textit{if } S_{\max} > Q_{\max} \end{cases} \tag{22}$$

The limits of segments are defined through equations (11)-(14).

Demonstration. The existence of equilibrium is ensured by condition (10) of proposition 6, and by the property of the density function on the supply side $\left(\int_0^{S_{\max}} \int_0^{Q_{\max}} \varphi(q, s)\mathrm{d}q\mathrm{d}s = 1\right)$ and the distribution of consumers groups weights on the demand side $\left(\sum_{i=1}^{N} \eta_i = 1\right)$.

The demonstration of the hedonic price function equations resulting from the equilibrium follows from the construction described above. □

3 Results and implications

3.1 Implicit prices of housing attributes

Theorem 2 establish the existence of equilibrium and its equations. The hedonic price function resulting from equilibrium is continuous and piecewise defined, and each segment corresponds to the equilibrium groups bid surfaces. However, the marginal hedonic price functions (represented implicit prices of housing attributes) are not continuous:

Theorem 3. *In the presence of an horizontal differentiation of housing, and discrete heterogeneity of consumer' preferences for housing attributes, both the implicit price of environmental quality and the implicit price of intrinsic attribute have discontinuities on the boundaries between the segments of the hedonic price function.*

Demonstration. See Appendix A.5. □

Theorem 3 extends the Baudry and Maslianskaia-Pautrel's (2012) result found with vertical differentiated housing supply: The segmentation of the hedonic price function is a result of discrete consumer heterogeneity whatever the differentiation on the supply side.

To illustrate the results of Theorem 3, we use numerical simulations computed for each set of single-crossing assumptions from the following nested CES utility function:

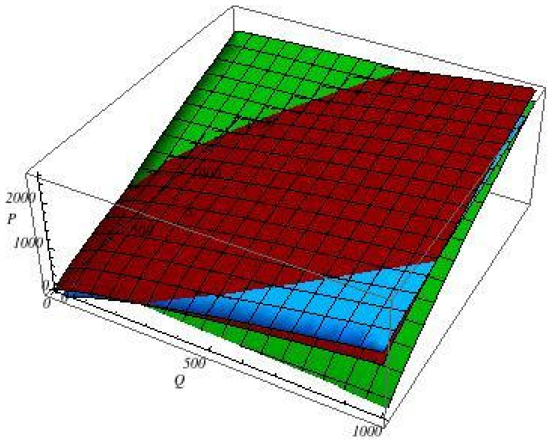
$$U(h(q,s|\beta_j),X) = [\alpha\ h(q,s|\beta_j)^\sigma + (1-\alpha)X^\sigma]^{\frac{1}{\sigma}}\ ,\ \sigma \in]0,1[,\ \alpha \in [0,1[. \tag{23}$$

The index of housing services level is: $h(q,s|\beta_j) = [\beta_j q^\sigma + (1-\beta_j)\, s^\sigma]^{\frac{1}{\sigma}}$.

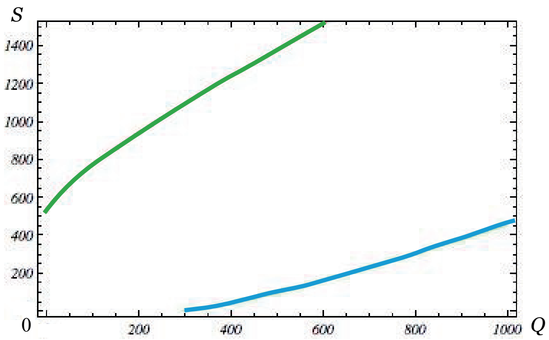
We study three groups of individuals according to their preferences parameter β with $\beta_1 = 0.1$ (low preference for the environmental housing quality), $\beta_2 = 0.5$ (average preference for the environmental housing quality), $\beta_3 = 0.7$ (strong preference for the environmental housing quality). The simulations are carried out for the consumer income $R = 19,900$ and the parameters of the utility function $\alpha = 0.4$ and $\sigma = 0.7$, calibrated from the French data. The group weights are respectively: $\eta_1 = 0.17, \eta_2 = 0.74, \eta_3 = 0.09$, and the joint distribution of housing attributes is supposed to be uniformly distributed on the intervals $[0, 1\ 000] \times [0, 1\ 500]$.

Figure 2 represents numerical simulations which illustrate the theoretical results. The hedonic price is the envelope surface of the groups' bid surfaces (figure 2(a)).[15] The graph 2(b) represents the boundaries between consumer groups in the plane (Q, S). The implicit price of environmental quality and the implicit price of intrinsic housing quality are both discontinuous on the borders of the segments (graphs 2(c) and 2(d)).

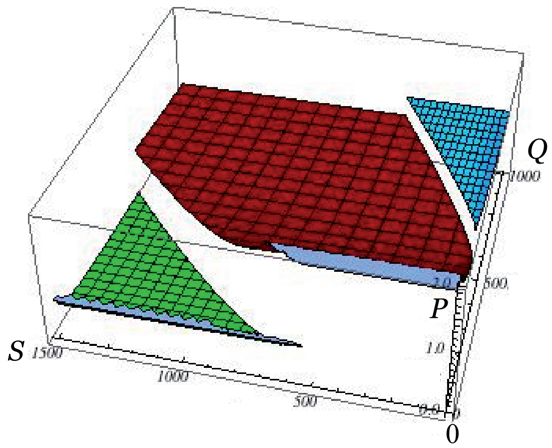
Figure 2: Hedonic price function and implicit prices. Application for a nested CES function



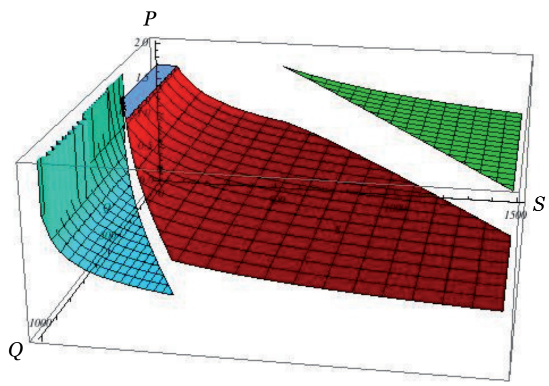
(a) Hedonic price function.



(b) Boundaries between groups (projection).



(c) Implicit price of the environmental housing quality.



(d) Implicit price of the intrinsic housing quality.

The generalized results on discontinuity of the implicit price is particularly important for environmental valuation. Indeed, it is contrary to Rosen's, which assumes at the second stage of the estimation procedure that the implicit price is continuous. Thus, our model is outside the scope of Rosen's model and the 2 stages estimation procedure cannot be used.

[15]The equilibrium equations are obtained in Appendix B.

3.2 Sorting of the demand

One of the most important implications of the hedonic equilibrium of the segmentation model concerns the sorting of the demand for environmental quality.

Proposition 7. *Under assumptions $H1$ and $H2$, the hedonic equilibrium leads to a partial sorting of demand for both housing attributes (see Figure 3):*
$\forall i \neq j$ (consumer groups), such that $\beta_i < \beta_j$ $\exists A \in i$, $B \in i$, $C \in j$ such that $Q_A < Q_C < Q_B$.

Proof. Follows from Theorem 2. □

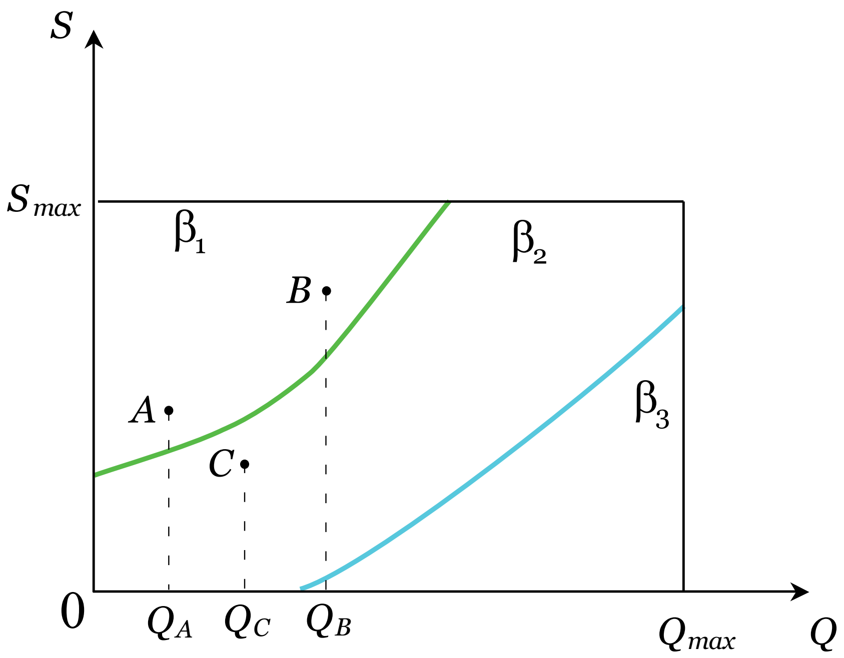
The results about partial sorting means that consumers with higher preference for environmental quality ($\beta_2 < \beta_1$ as shown at Figure 3) could chose at equilibrium housings with lower level of environmental quality than consumers with lower level of β ($Q_C < Q_B$). This result can be viewed as dual to the two dimensions stratification result obtained by Epple and Platt (1998) for a sorting model. In their model there is a partial stratification of consumers by their income and preferences at equilibrium, when consumers sort themselves into communities. Namely consumers with higher and lower income and/or preference parameters could be localized in the same community.

3.3 Implications for welfare analysis

Hedonic environmental valuation contributes to the analysis of public environmental policy by studying its impact on welfare. One measure of welfare is consumer surplus. The following section analyses implications of the present model for consumer surplus.

Consumer surplus for a group i can be defined as the difference between maximal consumer willingness to pay for housing exhibiting particular attributes and real expenditure on this housing. Maximal willingness to pay is given by the group's participation constraint, and the real expenditure corresponds to group's equilibrium bid function. So a group's consumer surplus can be written as: $CS_i = CP_i(q, s|\beta_i, u^{(CP_i)}) - E_i^*(q, s|\beta_i, u_i^*)$.

Figure 3: Partial sorting of housing attributes demand



If at equilibrium a group remains on its participation constraint it means that the sellers extract the totality of the consumer surplus for this group. If at equilibrium a group is on bid function other than its participation constraint, the group's consumers surplus is positive, because the price of housing is less than their maximal willingness to pay.

Proposition 8. *Under assumptions $H1$ and $H2$, $CS_{i^\circ} = 0$, where i° is found from (10).*

Proof. Follows Theorem 2. □

The question about which group fulfills its participation constraint depends on the relationship between the summary weight of the first groups and the share of the supply available for these groups to the left of the intersection of group participation constraints (see Theorem 2). Thus, in this case, the sellers do not necessarily extract the entire consumer surplus from the group with the lowest parameter of preferences β_1, contrarely to the case of vertical differentiated housing market. Therefore, a modification of the environmental quality of housing can involve a modification of the equilibrium and thus, the group from which the whole consumer surplus is extracted. This result can affect the valuation of the change in welfare following a modification of the environmental quality of housing, and therefore the environmental cost-benefit analysis.

4 Conclusion

The theoretical model developed in this paper shows how groupwise heterogeneity on the demand side influences the equilibrium of a housing market with horizontal differentiated supply and the formation of the hedonic price function. It develops on the hedonic analysis with segmentation of Baudry and Maslianskaïa-Pautrel (2012) for vertical differentiation of housing supply by investigating the more realistic case of horizontal differentiation. Both models appear to belong to a third type of modeling underlying hedonic environmental assessment complementing the " traditional" hedonic model of Rosen (1974) and "new" hedonic or sorting models, developed among others, by Epple and Platt (1998); Epple and Sieg (1999); Kuminoff (2009). While the first model considers the formation of an equilibrium in the market for differentiated products by assuming a continuous heterogeneity on the demand side and a continuous distribution on the supply side, and the second examines the allocation of a continuum of individuals with a continuous heterogeneity between a discrete number of communities (each characterized by an homogeneous amenities provision and housing prices within each community), our model examines the implications of discrete heterogeneity of consumers in the presence of a continuous distribution on the supply side.

Our modeling is based on the theoretical framework of Rosen (1974), assuming that the supply side distributions of two housing attributes (environmental quality and intrinsic quality) are fixed. This assumption is consistent with the short-term setting observed in empirical hedonic models. Each house is characterized by a level of housing services aggregating two housing attributes. We assume an horizontal differentiation of housing in which the aggregate function differs from one consumer group to another.

There are three main results. First, we demonstrate that the groupwise consumer heterogeneity leads to a segmentation of the hedonic price function in the housing market equilibrium.

Hedonic price function is continuous, but built of different segments corresponding to different consumer groups. As with vertical differentiation case, the segmentation of the hedonic price function implies a discontinuity of the implicit price of environmental quality of housing on the borders of the segments, so the "usual" estimation procedure developed in Rosen's model cannot be used.

Our second result concerns sorting of consumer demand. We demonstrate that the segmentation of hedonic price function leads to partial sorting of the demand for housing attributes by consumers groups. This result can be viewed as dual to the partial two-dimensional stratification by consumer income and preferences in sorting models (Epple and Platt, 1998).

Finally, our third result concerns consumer surplus extraction in equilibrium. Fixed supply and short-run assumption leads a subset of sellers to extract the total consumer surplus of one or more groups at equilibrium. We show that the group with the lowest parameter for environmental quality is not necessarily the one with its total surplus extracted. Recall that in the vertical differentiation case, the whole surplus is extracted from the first consumer group (with the lowest preference parameter). Thus segmentation has implications for the assessment of welfare related to changes in environmental quality, and thus for the cost/benefit environmental analysis. The study of these implications constitutes a promising topic for future research.

This article calls for further investigations. From the theoretical point of view, it would be interesting to study a mixed case with continuous consumers income heterogeneity and group-wise heterogeneity in housing attribute preferences. This would relax the strong assumption of equality of income among individuals in the presence of an horizontal differentiation of housing. In addition, since any decision on location is a joint result based on the " housing/work" choice, the segmentation model developed in this paper could be also extended to include the labor market in order to examine the formation of a "general hedonic equilibrium" as Kuminoff (2011) in a sorting model.

From an empirical point of view, the present paper develops the basic modeling for econometric methods which could accurately deal with aspects of the housing market segmentation. In particularly, the nonparametric models studied in Ekeland et al. (2004) and Heckman et al. (2010) seem to be able to deal better with the discreteness in the price function predicted by this paper.

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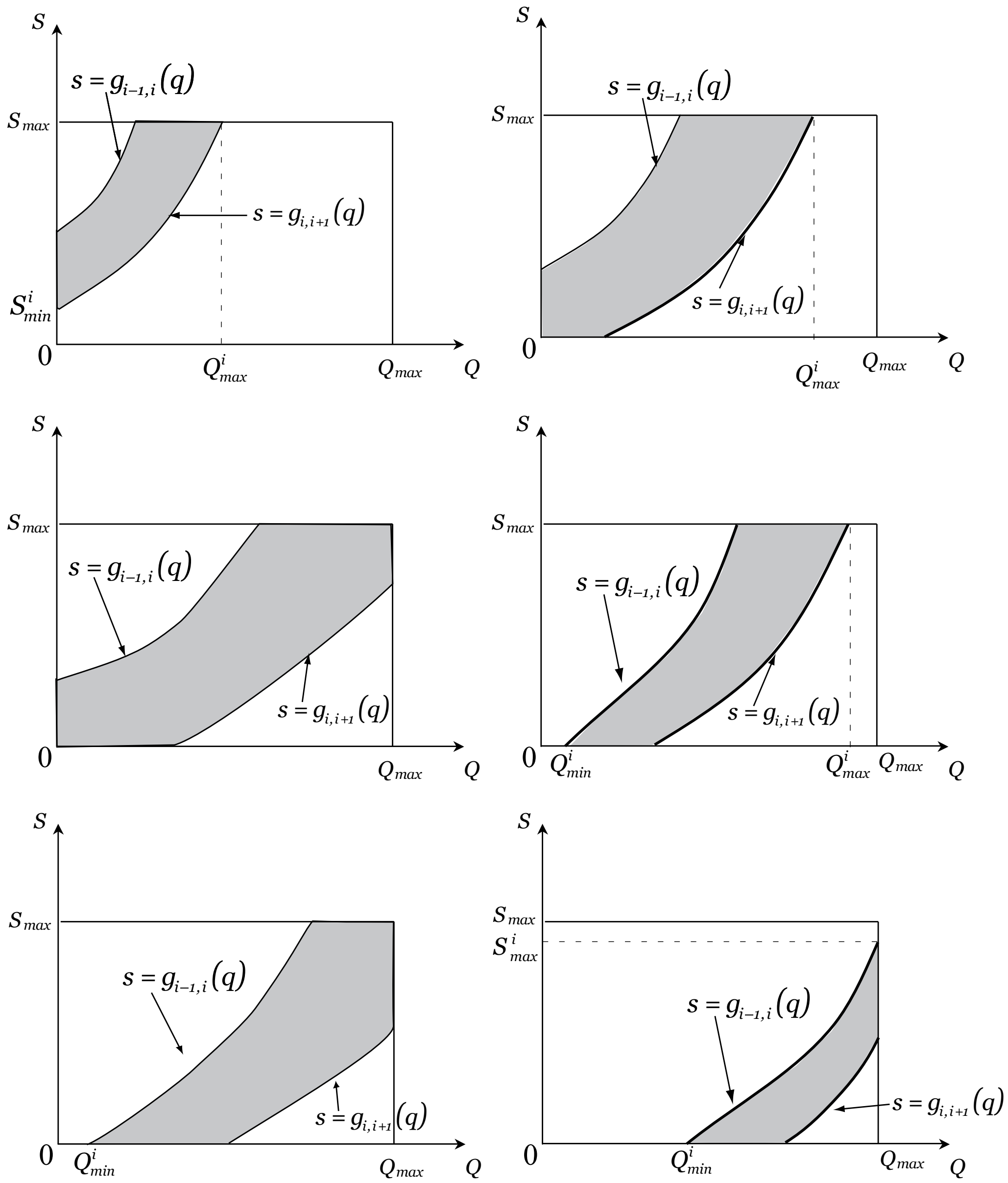
Figure 4: Different configurations for consumer groups boundaries



Appendices

A Proofs and demonstrations

A.1 Proof of proposition 3

Let two iso-H curves on the plane (Q, S), corresponding to the same housing services level $\tilde{h}$, of groups i and j such that $\beta_i < \beta_j$: $s = \tilde{s}_i(q|\beta_i) : h(q, \tilde{s}_i(q|\beta_i)|\beta_i) = \tilde{h}$ and $s = \tilde{s}_j(q|\beta_j) : h(q, \tilde{s}_j(q|\beta_j)|\beta_j) = \tilde{h}$. Let (q^*, s^*) be the intersection point of $\tilde{s}_i$ and $\tilde{s}_j$.

We assume that at equilibrium there exists $(q_i^{'}, s_i^{'}) \in \tilde{s}_i(\cdot)$ and $(q_j^{'}, s_j^{'}) \in \tilde{s}_j(\cdot)$ such that:

$$\begin{cases} q_j^{'} < q^* < q_i^{'} \\ s_j^{'} > s^* > s_i^{'} \end{cases} \tag{A.1}$$

This means that the iso-H curve of the group with higher level of β is located to the left of the iso-H curve of the group with lower level of β.

Since $\beta_i < \beta_j$, it follows that $s_j^{'} = \tilde{s}_j(q_j^{'}|\beta_j) > \tilde{s}_i(q_i^{'}|\beta_i) \doteq s_i^{''}$. Consequently, the point $(q_j^{'}, s_j^{'})$ is located above the iso-H curve $\tilde{s}_i(\cdot)$ of the group i. This means that individuals of group i achieve a higher utility level if they are located on the iso-H curve passing through the point $(q_j^{'}, s_j^{'})$ (the iso-H curves of both groups correspond to the same price level). The incentive constraint is not satisfied and thus the condition (A.1) is not satisfied at equilibrium.

Consequently, at equilibrium the iso-H curves are ordered in the plane (Q, S) in increasing order of β, from the left to the right along the axis Q. Q.E.D.

A.2 Demonstration of theorem 1

1°. The saturation of incentive constraints with respect to adjacent groups follows immediately from Proposition 3. Indeed, suppose that at equilibrium

$\left\{(\bar{Q}^*_{i,i+1}, \bar{S}^*_{i,i+1})\right\} = \left\{(q,s)\ :\ \left(h(q,s|\beta_i) = \bar{H}\right)\ \&\ \left(E_i^*(\bar{H}|\beta_i, u_i^*) = P^*(\bar{H})\right)\right\}$

$\bigcap \left\{(q,s)\ :\left(h(q,s|\beta_{i+1}) = \bar{H}\right)\ \&\ \left(E_{i+1}^*(\bar{H}|\beta_{i+1}, u_{i+1}^*) = P^*(\bar{H})\right)\right\}$. $(\bar{Q}^*_{i,i+1}, \bar{S}^*_{i,i+1})$ is the intersection point of the iso-H equilibrium curves of groups i and $i+1$, corresponding to a housing services level $\bar{H}$.

So, $h(\bar{Q}^*_{i,i+1}, \bar{S}^*_{i,i+1})|\beta_i) = h(\bar{Q}^*_{i,i+1}, \bar{S}^*_{i,i+1}|\beta_{i+1})$ and $E_i(\bar{Q}^*_{i,i+1}, \bar{S}^*_{i,i+1}|\beta_i, u_i^*) = E_{i+1}(\bar{Q}^*_{i,i+1}, \bar{S}^*_{i,i+1}|\beta_{i+1}, u_{i+1}^*)$.

Consequently $V(h(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1})|\beta_i), E_i(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1}|\beta_i,u^*_i))$

$= V(h(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1})|\beta_i), E_{i+1}(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1}|\beta_{i+1},u^*_{i+1}))$, and

$V(h(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1})|\beta_{i+1}), E_{i+1}(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1}|\beta_{i+1},u^*_{i+1})) = V(h(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1})|\beta_{i+1}), E_i(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1}|\beta_i,u^*_i))$,

from which $(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1}) \in CI^{k_{i,i+1}}_{i,i+1}$ and $(\bar{Q}^*_{i,i+1},\bar{S}^*_{i,i+1}) \in CI^{k_{ij}}_{i+1,i}$, so $CI^{k_{i+1,i}}_{i,i+1} \equiv E^*_{i+1}$ (the incentive constraint of the group i with respect to the group $i+1$ and the incentive constraint of the group $i+1$ with respect to group i are equilibrium bid surfaces of the groups i and $i+1$ respectively).

2°. The incentive constraint of group i with respect to group j such that: $j \neq i+1$, $j \neq i_1$ cannot be saturated at equilibrium because, if it were the iso-H curves of the groups between i and j would not be monotonic.

1° and 2° demonstrate the validity of theorem 1.

A.3 Proof of proposition 5

Consider the participation constraints of three groups $i_1 < i_2 < i_3$: $CP_{i_1} = E_{i_1}(s,q|\beta_{i_1},u^{(CP_{i_1})})$, $CP_{i_2} = E_{i_2}(s,q|\beta_{i_2},u^{(CP_{i_2})})$, $CP_{i_3} = E_{i_3}(s,q|\beta_{i_3},u^{(CP_{i_3})})$. Since the "outside the market" alternative does not depend on the parameter β, and by definition the participation constraint we obtain: $u^{(CP_{i_1})} = u^{(CP_{i_2})} = u^{(CP_{i_3})} = U(R,0) \doteq u^0$.

Adopting the following notation:

$s = f_{12}(q)$ for the intersection curve between the surfaces CP_{i_1} and CP_{i_2};

$s = f_{23}(q)$ for the intersection curve between the surfaces CP_{i_2} and CP_{i_3};

$s = f_{13}(q)$ for the intersection curve between the surfaces CP_{i_1} and CP_{i_2}.

By definition of the participation constraint

$$CP_{i_1}(q,f_{12}(q)|\beta_{i_1},u^0) = CP_{i_2}(q,f_{12}(q)|\beta_{i_2},u^0) = F_{12}(q,f_{12}(q)) \tag{A.2}$$

$$CP_{i_2}(q,f_{23}(q)|\beta_{i_2},u^0) = CP_{i_3}(q,f_{23}(q)|\beta_{i_3},u^0) = F_{23}(q,f_{23}(q)) \tag{A.3}$$

$$CP_{i_1}(q,f_{12}(q)|\beta_{i_1},u^0) = CP_{i_3}(q,f_{13}(q)|\beta_{i_3},u^0) = F_{13}(q,f_{13}(q)) \tag{A.4}$$

The single-crossing condition and the monotony of the boundary between two groups implies that either the curves $f_{12}(q)$, $f_{23}(q)$, $f_{13}(q)$ are similar, or each pair of curves has only one intersection point.

By definition of the "outside the market" alternative and of the participation constraint: $f_{12}(0) = f_{23}(0) = f_{13}(0) = 0$. Consequently, if the curves do not coincide they cross each other in the same point $(0, 0)$.

Let $f_{12}(q) \neq f_{23}(q) \neq f_{13}(q)$, if $q \neq 0$. We first suppose that $f_{12}(\tilde{q}) < f_{23}(\tilde{q}) < f_{13}(\tilde{q}), \forall \tilde{q} > 0$.

Adopting the following notation for a given level of q ($q = \tilde{q}$): $\tilde{s}_{12} = f_{12}(\tilde{q})$; $\tilde{s}_{23} = f_{23}(\tilde{q})$; $\tilde{s}_{13} = f_{13}(\tilde{q})$.

1°. By substituting the point $(\tilde{q}, \tilde{s}_{12})$ into (A.2), we obtain:

$$CP_{i_1}(\tilde{q}, \tilde{s}_{12}|\beta_{i_1}, u^0) = CP_{i_2}(\tilde{q}, \tilde{s}_{12}|\beta_{i_2}, u^0) = F_{12}(\tilde{q}, \tilde{s}_{12}) \doteq \tilde{F}_{12} \tag{A.5}$$

If $\tilde{h}_{12}$ for the level of H corresponding to iso-H curves of groups i_1 and i_2, which cross each other in the point $(\tilde{q}, \tilde{s}_{12})$, following (A.5), the price corresponding to the iso-H curve $\tilde{h}_{12}$ is $\tilde{F}_{12}$. Given the proof of proposition 3 (see Appendix A.1), the iso-H curve of the group i_1 is located to the left of the iso-H curve of the groups i_2.

Let (q'_{223}, s'_{223}) be the intersection point of the iso-H curve of the group i_2 ($h_{i_2}(q, s|\beta_{i_2}) = \tilde{h}_{12}$) due to the intersection curve of groups i_2 and i_3 participation constraints ($s = f_{23}(q)$). $E_{i_2}(q'_{223}, s'_{223}) = \tilde{F}_{12}$, because of the iso-H curve of the group i_2 corresponds to iso-price $\tilde{F}_{12}$.

Consider the iso-H curve of the group i_3, passes through the point (q'_{223}, s'_{223}). As this point belongs to the intersection curve of groups i_2 and i_3 participation constraints, it follows that: $E_{i_3}(q'_{223}, s'_{223}) = E_{i_3}(q'_{223}, s'_{223}) = \tilde{F}_{12}$.

As $f_{23}(q) > f_{12}(q)$, $\forall q > 0$, the iso-H curve of group i_2 is located to the right of the curve $f_{23}(q)$, which is contrary to the proof of proposition 3. Thus we obtain that $f_{23}(q) < f_{12}(q)$, $\forall q > 0$.

2°. Looking at the point $(\tilde{q}, \tilde{s}_{23})$. By substituting it into (A.3), we obtain $CP_{i_2}(\tilde{q}, \tilde{s}_{23}|\beta_{i_2}, u^0) = CP_{i_3}(\tilde{q}, \tilde{s}_{23}|\beta_{i_3}, u^0) = F_{23}(\tilde{q}, \tilde{s}_{23}) \doteq \tilde{F}_{23}$.

Let $(q^{'}_{212}, s^{'}_{212})$ be the intersection point of the iso-H curve of the group i_2 corresponding to the price level $\tilde{F}_{23}$, and of the intersection curve of the groups i_1 and i_2 participation constraints, $\tilde{f}_{12}$. Thus,

$$CP_{i_2}(q^{'}_{212}, s^{'}_{212}) = \tilde{F}_{23} \tag{A.6}$$

Let $(q^{'}_{312}, s^{'}_{312})$ be the intersection point of the iso-H curve of the group i_3 corresponding to the price level $\tilde{F}_{23}$, and of the intersection curve of the groups i_1 and i_2 participation constraints, $\tilde{f}_{12}$. We obtain for the iso-H curve of the group i_2 passing through the point $(q^{'}_{312}, s^{'}_{312})$

$$CP_{i_2}(q^{'}_{312}, s^{'}_{312}) = CP_{i_3}(q^{'}_{312}, s^{'}_{312}) = \tilde{F}_{23} \tag{A.7}$$

Conditions (A.6) and (A.7) mean that $CP_{i_2}(q^{'}_{212}, s^{'}_{212}) = CP_{i_2}(q^{'}_{312}, s^{'}_{312})$ or equivalently $f_{12}(q^{'}_{212}) = f_{12}q^{'}_{312}$, which contradicts the strict monotony of the function $s = f_{12}(q)$. Finally we obtain intersection curve $f_{12}(q)$ coincides with the intersection curve $f_{23}(q)$, $\forall q > 0$.

By proceeding in the same way we obtain $f_{12}(q) = f_{13}(q)$, and thus all participation constraints have the same intersection curve. Q.E.D.

A.4 Demonstration of proposition 6

Looking at the relationship between 1st group's weight, η_1, and the market share available for this group to the left of the curve $F(q, f(q))$, called $\Pi(f(q))$, two cases are possible:

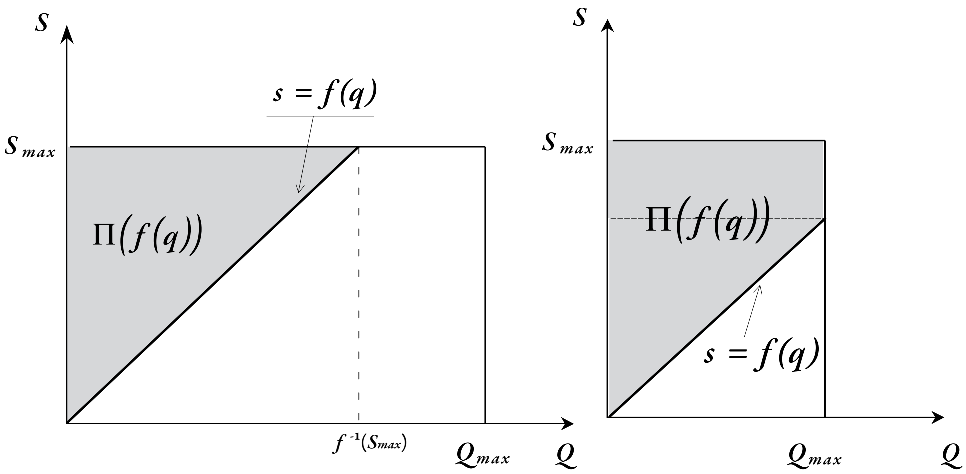
$$\text{Case (a):} \quad \Pi(f(q)) \leq \eta_1, \tag{A.8}$$

$$\text{Case (b):} \quad \Pi(f(q)) > \eta_1, \tag{A.9}$$

where the share $\Pi(f(q))$ is defined according to whether the intersection curve $s = f(q)$ achieves the maximum level of S inside or outside the domain of the variable Q (see figure 5):

$$\Pi(f(q)) = \left[\begin{array}{l} \int_0^{S_{\max}} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \quad \text{si } S_{\max} \leq Q_{\max}; \\ \\ \int_0^{f(Q_{\max})} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s + \\ + \int_{f(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \quad \text{si } S_{\max} > Q_{\max} \end{array} \right.$$

Figure 5: Two cases for market share $\Pi(f(q))$ calculus



Case (a). The market share $\Pi(f(q))$ is lower that the 1^{st} group's weight

The condition (A.8) implies that consumers of the 1^{st} group are located at equilibrium on their participation constraint CP_1. According to (A.8) and (5) the boundary between group 1 and group 2 is located to the right of the curve of intersection of participatory constraints, i.e. where the groups' participation constraints are ordered in increasing order of the parameter β. In this case, if at equilibrium the individuals of the 1^{st} group are on their participation constraint, consumers in other groups can not even consider the purchase of goods purchased by the consumers of the 1^{st} group on the left of the intersection because their participation constraints are below that one of the 1^{st} group. The boundary between groups 1 and 2 in plane (Q, S) is defined using the following equation: $\int_0^{S^1_{\max}} \int_0^{g_{12}^{-1}(s)} \varphi(q, s)\,\mathrm{d}q\,\mathrm{d}s = \eta_1$.[16]

Proceeding in a similar way to assumptions $(H^{'}1)$ and $(H^{'}2)$, (where the intersection of participation constraints belongs to the plane of coordinates (S, P))we obtain that at equilibrium, consumers of groups $2, ..., N$ are located on the incentive constraints of their group with respect to the previous group and the boundary $s = g_{i-1,i}(q)$ between two groups i and $i-1$ is defined implicitly from the equation $\sum_{k=1}^{i-1} \Pi_k(g_{k,k+1}(q)) = \sum_{k=1}^{i-1} \eta_k,\ i \geq 2$; where the sum of the market shares $\sum_{k=1}^{i-1} \Pi_k(g_{k,k+1}(q))$ is defined according to the disposition of the boundary between two

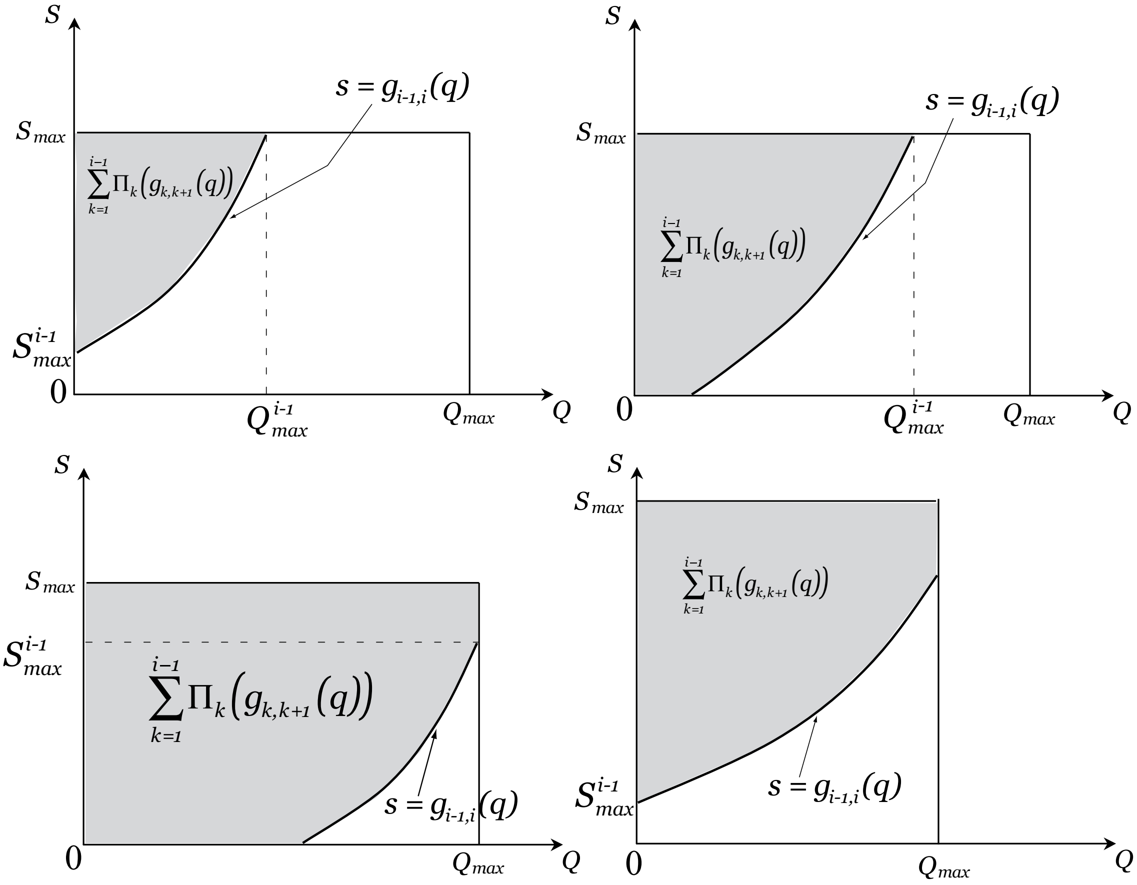
[16]If the condition (A.8) is an equality, it means that at equilibrium the group 2 participation constraint is also saturated, and starting from the group 3 the incentive constraints with the previous group are saturated.

groups $s = g_{i-1,i}(q)$ with respect to the domains of variables Q and S (see figure 6):

$$\sum_{k=1}^{i-1} \Pi_k(g_{k,k+1}(q)) = \left[\begin{array}{l} \int_{S^{i-1}_{\min}}^{S_{\max}} \int_{Q^{i-1}_{\min}}^{g^{-1}_{i-1,i}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \quad \text{si } S_{\max} \le Q_{\max}, \\ \\ \int_{S^{i-1}_{\min}}^{g_{i-1,i}(Q_{\max})} \int_{Q^{i-1}_{\min}}^{g^{-1}_{i-1,i}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s \\ \\ + \int_{g_{i-1,i}(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \quad \text{si } S_{\max} > Q_{\max}; \end{array}\right.$$

where $\begin{cases} S^{i-1}_{\min} = g_{i-1,i}(0), \\ Q^{i-1}_{\min} = 0, \end{cases}$ si $g_{i-1,i}(0) \ge 0$, and $\begin{cases} S^{i-1}_{\min} = 0, \\ Q^{i-1}_{\min} = g^{-1}_{i-1,i}(0), \end{cases}$ si $g^{-1}_{i-1,i}(0) \ge 0$.

Figure 6: Calculus of $\sum_{k=1}^{i-1} \Pi_k(g_{k,k+1}(q))$



Case (b). The market share $\Pi(f(q))$ is higher than the 1^{st} group weight

Condition (A.9) implies that at equilibrium, consumers of the first group can not be situated on their constraint participation CP_1. Indeed, if they could, the boundary between the groups 1 and 2 would be located to the left of the intersection curve $F(q, f(q))$, where groups' participation constraints are ordered in inverse order to preference parameters β. Thus the surface CP_1 would be above the surface CP_2 on the corresponding points on the frontier between the two groups. The consumers of group 2 can not be located above the surface CP_2, so if the situation described above is an equilibrium, then the prices of homes purchased by group 2 would be located below the equilibrium bid surface of group 1, which is contrary to proposition 2.

In contrast, consumers of group 2 may be located on their constraint participation CP_2 at equilibrium, subject to the boundary between groups 2 and 3 being located to the right of the intersection of participation constraints:

$$\eta_1 + \eta_2 \geq \Pi(f(q)) \tag{A.10}$$

$$\Pi(f(q)) = \left[\begin{array}{l} \int_0^{S_{\max}} \int_0^{f^{-1}(s)} \varphi(q,s)\, \mathrm{d}q\, \mathrm{d}s, \quad \text{if } S_{\max} \leq Q_{\max}; \\ \\ \int_0^{f(Q_{\max})} \int_0^{f^{-1}(s)} \varphi(q,s)\, \mathrm{d}q\, \mathrm{d}s \;+\; \int_{f(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\, \mathrm{d}q\, \mathrm{d}s, \quad \text{if } S_{\max} > Q_{\max} \end{array} \right.$$

In this case, the group 1 consumers are located at equilibrium on the bid surface which is the incentive constraint of group 1 with respect to group 2, passing through the boundary between groups 1 and 2. The function of hedonic price equilibrium for groups 3, 4,..., I, is defined as in the case (a): on the basis of the incentive constraints of the group relative to the previous group, passing through boundary between the groups.

Let $s = g_{12}(q)$ be the equilibrium boundary between groups 1 and 2 (projection on the plane (Q,S)), $s = g_{23}(q)$ the equilibrium boundary between the groups 2 and 3,..., $s = g(i, i+1)$ the equilibrium boundary between the groups i and $i+1$. So the equilibrium hedonic price function is composed on segments $E_1^*, E_2^*, E_3^*, ..., E_N^*$, such that the following equations are satisfied:

$E_2^* = CP_2(q, s|\beta_2, u_2^*)$,

$V\left(h(q,s|\beta_1), E_1^*\left(q,s|\beta_1,u_1^*\right)\right) = V\left(h(q,g_{12}(q)|\beta_1), E_2^*\left(q,g_{12}(q)|\beta_2,u_2^*\right)\right)$

$V\left(h(q,s|\beta_i), E_i^*\left(q,s(q)|\beta_i,u_i^*\right)\right) = V\left(h(q,g_{i-1,i}(q)|\beta_i), E_{i-1}^*\left(q,g_{i-1,i}(q)|\beta_{i-1},u_{i-1}^*\right)\right), i \in \{3,...,I\}$

$\forall\, i \in \{1,...,I-1\}$, $s = g_{i,i+1}(q) : E_i^*(q, g_{i,i+1}|\beta_i, u_i^*) = E_{i+1}^*(q, g_{i,i+1}|\beta_{i+1}, u^{i+1})$,

$\sum_{k=1}^{i} \Pi_i(g_{i,i+1}(q)) = \sum_{k=1}^{i} \eta_i,\ i \in \{1,...,I\}$

$$\sum_{k=1}^{i} \Pi_i(g_{i,i+1}(q)) = \left[\begin{array}{l} \int_{S^i_{\min}}^{S_{\max}} \int_{Q^i_{\min}}^{g^{-1}_{i,i+1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \quad \text{si } S_{\max} \le Q_{\max}; \\ \\ \int_{S^i_{\min}}^{g_{i,i+1}(Q_{\max})} \int_{Q^i_{\min}}^{g^{-1}_{i,i+1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s \\ \\ +\int_{g_{i,i+1}(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \text{ si } S_{\max} > Q_{\max} \end{array} \right.$$

The segments limits are defined according to the disposition of the boundaries on the left and on the right of each segment (see Figure 4 which illustrates different possible configurations). Thus, on each interior segment, $\forall\ 1 < i < I$, $q \in \left[Q^i_{\min}, Q^i_{\max}\right]$, $s \in \left[S^i_{\min}, S^i_{\max}\right]$:

$$Q^i_{\min} = \left[\begin{array}{l} 0, \ \text{ if } g_{i-1,i}(0) \ge 0, \\ g^{-1}_{i-1,i}(0), \ \text{ if } g^{-1}_{i-1,i}(0) \ge 0; \end{array} \right. \qquad Q^i_{\max} = \left[\begin{array}{ll} g^{-1}_{i,i+1}(S_{\max}), & \text{if } g_{i,i+1}(Q_{\max}) \ge S_{\max}, \\ Q_{\max}, & \text{if } g_{i,i+1}(Q_{\max}) < S_{\max}. \end{array} \right.$$

$$S^i_{\min} = \left[\begin{array}{l} g_{i,i+1}(0), \ \text{ if } g_{i,i+1}(0) \ge 0, \\ 0, \ \text{ si } g^{-1}_{i,i+1}(0) \ge 0; \end{array} \right. \qquad S^i_{\max} = \left[\begin{array}{ll} S_{\max}, & \text{if } g_{i-1,i}(Q_{\max}) \ge S_{\max}, \\ g_{i-1,i}(Q_{\max}), & \text{if } g_{i-1,i}(Q_{\max}) < S_{\max}. \end{array} \right.$$

For the first segment, $i = 1$, $q \in \left[0, Q^1_{\max}\right]$, $s \in \left[S^1_{\min}, S_{\max}\right]$:

$$Q^1_{\max} = \left[\begin{array}{ll} g^{-1}_{1,2}(S_{\max}), & \text{if } g_{1,2}(Q_{\max}) \ge S_{\max}, \\ Q_{\max}, & \text{if } g_{1,2}(Q_{\max}) < S_{\max}. \end{array} \right. \qquad S^1_{\min} = \left[\begin{array}{l} g_{1,2}(0), \ \text{ if } g_{1,2}(0) \ge 0, \\ 0, \ \text{ if } g^{-1}_{1,2}(0) \ge 0; \end{array} \right.$$

For the last segment, $i = I$, $q \in \left[Q^N_{\min}, Q_{\max}\right]$, $s \in \left[0, S^N_{\max}\right]$:

$$Q^N_{\min} = \left[\begin{array}{ll} g^{-1}_{N-1,N}(0), & \text{if } g^0_{N-1,N}(0) \ge 0, \\ 0, & \text{if } g_{N-1,N}(0) \ge 0. \end{array} \right. \qquad S^N_{\max} = \left[\begin{array}{l} S_{\max}, \ \text{ if } g_{N-1,N}(Q_{\max}) \ge S_{\max}, \\ g_{N-1,N}(Q_{\max}), \ \text{ if } g_{N-1,N}(Q_{\max}) < S_{\max}; \end{array} \right.$$

Recall the condition (A.10) necessary for the equilibrium described above $\eta_1 + \eta_2 \ge \Pi(f(q))$, where $\Pi(f(q)) = \left[\begin{array}{l} \int_0^{S_{\max}} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \ \text{ if } S_{\max} \le Q_{\max}; \\ \int_0^{f(Q_{\max})} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s + \\ \int_{f(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, \ \text{ if } S_{\max} > Q_{\max} \end{array} \right.$

This condition can be generalized for all groups and finally we obtain: the group i° which saturates its participation constraint at equilibrium can be found using the following condition:

$$i^\circ = \min_{i \in \{1,\ldots,I\}} i \ : \ \sum_{k=1}^{i} \eta_k \ge \Pi(f(q)). \text{ Q.E.D.}$$

A.5 Demonstration of theorem 3

Looking at the section of the hedonic price surface corresponding to a group's i equilibrium bid function. By the definition 1 of the individual bid function, the equilibrium prices for the group i satisfy:

$$V(h(q,s), P^*(q,s)) = u_i^*, \quad (q,s) \in Dh_i \tag{A.11}$$

where u_i^* is the equilibrium reference utility of group i, and Dh_i is its definition domain equal to $[Q_i^{\min}, Q_i^{\max}] \times [S_i^{\min}, S_i^{\max}]$.

The partial derivatives of (A.11) with respect to Q, and with respect to S, give:

$$\left.\frac{\partial P}{\partial Q}\right|_{(q,s)\in Dh_i} = -\left.\frac{\partial V/\partial h\ \partial h/\partial Q}{\partial V/\partial P}\right|_{(q,s)\in Dh_i}, \tag{A.12}$$

$$\left.\frac{\partial P}{\partial S}\right|_{(q,s)\in Dh_i} = -\left.\frac{\partial V/\partial h\ \partial h/\partial S}{\partial V/\partial P}\right|_{(q,s)\in Dh_i}. \tag{A.13}$$

The equations (A.12) and (A.13) are verified for each group $i \in \{1, ..., I\}$.

We now study two adjacent groups i and $i+1$. The Spence-Mirrlees conditions $(H1)$ and $(H2)$, imply that $\frac{\partial}{\partial \beta}\left(\frac{\partial P}{\partial Q}\right) > 0$, $\forall S$, $\forall \beta$, and $\frac{\partial}{\partial \beta}\left(\frac{\partial P}{\partial S}\right) < 0$, $\forall Q$, $\forall \beta$. Consequently:

$$\left.\frac{\partial P_i^*}{\partial Q}\right|_{q=Q_i^{\max}, s=g_{i,i+1}(q)} < \left.\frac{\partial P_{i+1}^*}{\partial Q}\right|_{q=Q_i^{\max}, s=g_{i,i+1}(q)}, \tag{A.14}$$

$$\left.\frac{\partial P_i^*}{\partial S}\right|_{q=Q_i^{\max}, s=g_{i,i+1}(q)} > \left.\frac{\partial P_{i+1}^*}{\partial S}\right|_{q=Q_i^{\max}, s=g_{i,i+1}(q)}, \tag{A.15}$$

as $\beta_i < \beta_{i+1}$.

The inequalities (A.14) and (A.15) involve the discontinuity of implicit prices of housing environmental quality and of housing intrinsic quality on the boundaries of the segments. This demonstrates the theorem A.5. Q.E.D.

B Application to a nested CES direct utility function

In the case of a nested CES direct utility function the equation of group i participation constraint, CP_i, becomes $[\alpha\,(\beta_i q^\sigma + (1-\beta_i)s^\sigma) + (1-\alpha)(R - CP_i(q,s))^\sigma]^{\frac{1}{\sigma}} = (1-\alpha)^{\frac{1}{\sigma}} R$, $CP_i(q,s) = R - \left[R^\sigma - \frac{\alpha}{1-\alpha}\ (\beta_i q^\sigma + (1-\beta_i)s^\sigma)\right]^{\frac{1}{\sigma}}$. The solution of the equation $CP_i(q,s) = CP_j(q,s)$ provides the equation for the intersection curve of participation constraints $CP_i(q,s)$ and $CP_j(q,s)$:

$$s = q. \tag{B.1}$$

According to the theoretical result, the intersection function is increasing in q and does not depend on the β.

The group i equilibrium bid function for the nested CES specification case is written $E_i^*(q,s|\beta_i,u_i^*) = R - \left[\frac{1}{1-\alpha}\,(u_i^*)^\sigma - \frac{\alpha}{1-\alpha}\,(\beta_i q^\sigma + (1-\beta_i)s^\sigma)\right]^{\frac{1}{\sigma}}$, where u_i^* is the reference utility level in the bid function definition.

According to proposition 6, the group for which the participation constraint is fulfilled, is obtained from: $\sum_{k=1}^{i^\circ} \eta_k \geq \Pi(f(q))$, where $\Pi(f(q)) = \begin{cases} \int_0^{S_{\max}} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, & \text{if } S_{\max} \leq Q_{\max}; \\ \int_0^{f(Q_{\max})} \int_0^{f^{-1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s \\ + \int_{f(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, & \text{if } S_{\max} > Q_{\max}. \end{cases}$.

In the case of an uniform joint density function by replacing $f^{-1}(s)$ by the equation (B.1), the condition is written $\sum_{k=1}^{i^\circ} \eta_k \geq \begin{cases} \frac{S_{\max}}{2\,Q_{\max}}, & \text{if } S_{\max} \leq Q_{\max}; \\ 1 - \frac{Q_{\max}}{2\,S_{\max}}, & \text{if } S_{\max} > Q_{\max} \end{cases}$.

One the group i° which fulfills its participation constraint is defined, the hedonic price function corresponds on the interval $\left[Q_{\min}^{i^\circ}, Q_{\max}^{i^\circ}\right] \times \left[S_{\min}^{i^\circ}, S_{\max}^{i^\circ}\right]$ to the participation constraint of this group: $P_{i^\circ}^* = E_{i^\circ}^* = CP_{i^\circ} = R - \left[R^\sigma - \frac{\alpha}{1-\alpha}\,(\beta_{i^\circ} q^\sigma + (1-\beta_{i^\circ})s^\sigma)\right]^{\frac{1}{\sigma}}$, $q \in \left[Q_{\min}^{i^\circ}, Q_{\max}^{i^\circ}\right]$, $s \in \left[S_{\min}^{i^\circ}, S_{\max}^{i^\circ}\right]$, where the segment limits are defined from the equations (11)-(14).

At equilibrium according to the theorem 2, for the consumers of groups i, such that $i < i^\circ$, the hedonic price function corresponds to their incentive constraints with respect to the next group, passing through the boundary with the next group. The equation of the boundary, $s = g_{i,i+1}(q)$ is obtained from the equation (20): $s = \left(q^\sigma - \frac{(u_{i+1}^*)^\sigma - (u_i^*)^\sigma}{\alpha(\beta_{i+1}-\beta_i)}\right)^{\frac{1}{\sigma}}$, and the incentive constraint of group i with respect to the group $i+1$ (equation (18) of theorem 2) becomes in this case: $[\alpha\,(\beta_i q^\sigma + (1-\beta_i)s^\sigma) + (1-\alpha)(R - E_i(q,s|\beta_i,u_i^*))^\sigma]^{\frac{1}{\sigma}}$

$= \left[\alpha\,(\beta_i \tilde{q}^\sigma + (1-\beta_i)\tilde{s}^\sigma) + (1-\alpha)(R - E_{i+1}(\tilde{q},\tilde{s}|\beta_{i+1},u_{i+1}^*))^\sigma\right]^{\frac{1}{\sigma}}$, $\tilde{s} = g_{i,i+1}(\tilde{q}) = \left(\tilde{q}^\sigma - \frac{(u_{i+1}^*)^\sigma - (u_i^*)^\sigma}{\alpha(\beta_{i+1}-\beta_i)}\right)^{\frac{1}{\sigma}}$

from which

$$E_i(q,s) = R - \left[\left(R - E_{i+1}(\tilde{q},\tilde{s}|\beta_{i+1},u_{i+1}^*)\right)^\sigma - \frac{\alpha}{1-\alpha}\left(\beta\, q^\sigma + (1-\beta_i)s^\sigma + \tilde{q}^\sigma + (1-\beta_i)\,\frac{(u_i^*)^\sigma - (u_{i+1}^*)^\sigma}{\alpha(\beta_{i+1}-\beta_i)}\right)\right]^{\frac{1}{\sigma}}.$$

For the consumers of groups $i > i^\circ$ the equilibrium housing price corresponds to their incentive constraints with respect to the previous group. The equation of the boundary is written in this case: $s = \left(q^\sigma - \frac{(u_i^*)^\sigma - (u_{i-1}^*)^\sigma}{\alpha(\beta_i - \beta_{i-1})}\right)^{\frac{1}{\sigma}}$ and the incentive constraint (equation 19 of the theorem 2) becomes: $[\alpha\,(\beta_i q^\sigma + (1-\beta_i)s^\sigma) + (1-\alpha)(R - E_i(q,s|\beta_i,u_i^*))^\sigma]^{\frac{1}{\sigma}}$

$= \left[\alpha \left(\beta_i \tilde{q}^{\sigma} + (1-\beta_i)\tilde{s}^{\sigma}\right) + (1-\alpha)(R - E_{i-1}(\tilde{q},\tilde{s}|\beta_{i-1}, u^*_{i-1}))^{\sigma}\right]^{\frac{1}{\sigma}}$, $\tilde{s} = g_{i,i-1}(\tilde{q}) = \left(\tilde{q}^{\sigma} - \frac{(u^*_i)^{\sigma} - (u^*_{i-1})^{\sigma}}{\alpha(\beta_i - \beta_{i-1})}\right)^{\frac{1}{\sigma}}$

for which

$$E_i(q,s) = R - \left[\left(R - E_{i-1}(\tilde{q},\tilde{s}|\beta_{i-1}, u^*_{i-1})\right)^{\sigma} - \frac{\alpha}{1-\alpha}\left(\beta_i\, q^{\sigma} + (1-\beta_i)s^{\sigma} + \tilde{q}^{\sigma} - (1-\beta_i)\,\frac{(u^*_i)^{\sigma} - (u^* i-1)^{\sigma}}{\alpha(\beta_i - \beta_{i-1})}\right)\right]^{\frac{1}{\sigma}}.$$

The reference utility levels $\{u^*_i\}_{i=1}^N$ are obtained from conditions (21)-(22) of the equality of market share of a group and its weight (cf. 22). In the case of the nested CES utility function and joint uniform density these conditions are written as:

$$\sum_{k=1}^{i} \eta_k = \left[\begin{array}{ll} \int_{S^i_{\min}}^{S_{\max}} \int_{Q^i_{\min}}^{g^{-1}_{i,i+1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, & \text{if } S_{\max} \le Q_{\max}; \\ \int_{S^i_{\min}}^{g_{i,i+1}(Q_{\max})} \int_{Q^i_{\min}}^{g^{-1}_{i,i+1}(s)} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s & \\ + \int_{g_{i,i+1}(Q_{\max})}^{S_{\max}} \int_0^{Q_{\max}} \varphi(q,s)\,\mathrm{d}q\,\mathrm{d}s, & \text{if } S_{\max} > Q_{\max}, \end{array}\right. \tag{B.2}$$

$i \in \{1, ..., I\}$

$s = g_{i,i+1}(q) = \left(q^{\sigma} - \frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}}$, $\varphi(q,s) = \frac{1}{S_{\max} Q_{\max}}$,

$Q^i_{\min} = \left[\begin{array}{l} 0, \ \text{si } g_{i,i+1}(0) \ge 0, \\ g^{-1}_{i,i+1}(0), \ \text{if } g^{-1}_{i,i+1}(0) \ge 0; \end{array}\right.$, $S^i_{\min} = \left[\begin{array}{l} g_{i,i+1}(0), \ \text{if } g_{i,i+1}(0) \ge 0, \\ 0, \ \text{si } g^{-1}_{i,i+1}(0) \ge 0. \end{array}\right.$

By construction of the equilibrium, $u^*_i < u^*_{i+1}$, if $i^{\circ} < i \le I-1$, and $u^*_i > u^*_{i+1}$, if $1 \le i < i^{\circ}$.

Consequently:

A. For the groups on the right of the group fulfills its participation constraint, $i^{\circ} < i \le I-1$, $g^{-1}_{i,i+1}(0) = \left(\frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} > 0$, so the expression (B.2) becomes:

A.1. If $S_{\max} \le Q_{\max}$, $\sum_{k=1}^{i} \eta_k = \frac{1}{S_{\max} Q_{\max}} \int_0^{S_{\max}} \left(s^{\sigma} + \frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} \mathrm{d}s - \frac{1}{Q_{\max}} \left(\frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}}$

A.2. If $S_{\max} > Q_{\max}$, $\sum_{k=1}^{i} \eta_k = \frac{1}{S_{\max} Q_{\max}} \int_0^{g_{j,j+1}(Q_{\max})} \left(s^{\sigma} + \frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} \mathrm{d}s \ +1$
$- \frac{1}{Q_{\max}} \left(\frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} - \frac{1}{S_{\max}} \left(Q^{\sigma}_{\max} - \frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}}$

B. For the groups on the left from the group fulfills its participation constraint, $1 \le i < i^{\circ}$, $g_{i,i+1}(0) = \left(\frac{(u^*_i)^{\sigma} - (u^*_{i+1})^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} > 0$, so the expression (B.2) becomes:

B.1. If $S_{\max} \le Q_{\max}$, $\sum_{k=1}^{i} \eta_k = \frac{1}{S_{\max} Q_{\max}} \int_{S^i_{\min}}^{S_{\max}} \left(s^{\sigma} - \frac{(u^{(i)})^{\sigma} - (u^{(i+1)})^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} \mathrm{d}s$, where $S^i_{\min} = \left(\frac{(u^*_i)^{\sigma} - (u^*_{i+1})^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}}$

B.2. If $S_{\max} > Q_{\max}$, $\sum_{k=1}^{i} \eta_k = \frac{1}{S_{\max} Q_{\max}} \int_{S^i_{\min}}^{g_{j,j+1}(Q_{\max})} \left(s^{\sigma} - \frac{(u^*_i)^{\sigma} - (u^*_{i-1})^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}} \mathrm{d}s +$
$+1 - \frac{1}{S_{\max}} \left(Q^{\sigma}_{\max} - \frac{(u^*_{i+1})^{\sigma} - (u^*_i)^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}}$, where $S^i_{\min} = \left(\frac{(u^*_i)^{\sigma} - (u^*_{i+1})^{\sigma}}{\alpha(\beta_{i+1} - \beta_i)}\right)^{\frac{1}{\sigma}}$